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Welfare Characterization of Monetary-Applied Models and Three Implications*

Samuel de Abreu Pessoa[†]

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Abstract

This paper demonstrates that the applied monetary models - the Sidrauski-type models and the cash-in-advance models, augmented with a banking sector that supplies money substitutes services - imply trajectories which are Pareto-Optimum restricted to a given path of the real quantity of money. As a consequence, three results follow: First, Bailey's formula to evaluate the welfare cost of inflation is indeed accurate, if the long-run capital stock does not depend on the inflation rate and if the compensate demand is considered. Second, the relevant money demand concept for this issue - the impact of inflation on welfare - is the monetary base. Third, if the long-run capital stock depends on the inflation rate, this dependence has a second-order impact on welfare, and, conceptually, it is not a distortion from the social point of view. These three implications moderate some evaluations of the welfare cost of the perfect predicted inflation.

Key words: Money; Inflation; Welfare; Financial Services

JEL Classification: E31; O42

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1 Introduction

This paper establishes that the trajectories implied by the applied monetary models - the Sidrauski and the cash-in-advance class of models - are Pareto-Optimum restricted for a given path of the real quantity of money. Although these models do not present Pareto-Optimum solutions, the unique mode of improving the welfare is to increase individual money holdings. A Social Planner who can not stimulate individuals to increase their money holdings will do no better than the market. This property is general and applies to both families of monetary models augmented to take into consideration a banking sector, which provides services that are substitutes for money services. This last class of models displays the observable phenomenon of the inflow of production factors, capital and labor, and, consequently, the increase of the share in the product of the banking sector, along with the inflation rate. This property of welfare has three implications.

The **first** refers to the measurement of the welfare cost of inflation. Since Bailey's (1956) classic paper, economists have been accustomed to measuring the welfare cost of perfectly foreseen inflation by the area under the inverse money demand. Notwithstanding, there has not been much effort to attempt to gather a more solid theoretical foundation for this approach. It is shown that Bailey's formula is a general equilibrium measure and a corollary of the welfare property of these models, if the economy displays long-run capital's neutrality and if the compensate money demand is considered.

The **second** consequence of the welfare characterization is that the relevant concept of money, as far as the impact of inflation on welfare is concerned, is the narrow monetary aggregate, the monetary base. Money is the good which has private but not social cost. In this specific sense the demand deposit should be excluded from the concept of money.¹ It is offered by the banking institutions, and, consequently, has a positive social cost. To the best of my knowledge, it seems that this point has not been attracting the deserved attention by the monetary theorists. Bailey's discussion is not very clear in this respect. He begins his paper supposing that banks are not present. Afterwards, he introduces banks.² According to Bailey, if banks work rationally, then the correct concept is the monetary base; otherwise, the M_1 demand should be considered, although it is not very clear what he means by a bank

¹The issue here is not the liquidity of inside money vis-a-vis outside money. As far as liquidity is concerned, it seems to me that both should be classed as money.

²See Bailey (1956), p. 103 and 104.

not “behaving absolutely rationally.” Lucas (1981b), Cooley and Hansen (1990), and Lucas (2000), employ M_1 ; Barro (1972), Fischer (1981), Pastore (1994), and Aiyagari, Braun and Eckstein (1998), use M_0 . Given that usually the monetary multiplier is a number between 2 and 3, those calculations of the welfare cost of inflation that employ M_1 are overvaluing it by a factor between 2 and 3.

Since the work of Stockman (1981), it is known that if the cash-in-advance restriction applies to investment, the long-run capital stock will depend on the inflation rate. Because “Inflation acts as a tax on investment even in the absence of explicit taxation,”³ the higher the inflation is, the lower the steady-state capital stock is. The **third** outcome of the welfare characterization brought about by this paper is to show that from the point of view of welfare, this distortion is totally different from an explicit taxation on capital. In this last case, the distortion produces an edge between the social value of capital and the private one, or, in other words, there is a first-order impact of variations of the capital-accumulation path on welfare. For the former case, it is shown that variations on the capital path do not have a first-order impact on welfare. As a consequence, the paper argues that the distortion effect of inflation on welfare, if the transition dynamic is considered, is second-order small.

Notwithstanding the importance of these classes of models, there is not a sharp characterization of their welfare properties. The objective in this paper is to establish under which conditions the Central Planner’s solution is equal to the market solution, or, saying differently, what sort of restrictions should be placed on the Central Planner to reproduce the market economy. In this sense, this paper is complementary to Cole and Kocherlakota (1998). Their concern is to determine under which conditions the market economy replicates the unrestricted command economy and which policies sustain this path.

Two recent contributions to the topic of Inflation and Welfare, in the tradition of Bailey’s paper, are Aiyagari, Braun, and Eckstein’s (1998) and Lucas’s (2000).⁴ The main difference between the formulation accomplished in this paper and Lucas’s paper is the specific way the impact of inflation on welfare is calculated. Lucas evaluated it by the proportional increase in consumption, which makes the household indifferent between the two situations - in the presence of or without inflation. In this paper, the welfare cost of inflation is defined as the income which should be given to the household in order to compensate her by the harm

³Stockman (1981), p. 391.

⁴Another recent contribution is English (1999), although his main interest is inflation and banking sector output’s increase.

caused by the inflation. Additionally, Lucas does not consider the existence of a banking sector which supplies money substitutes services, and, consequently, his analysis cannot distinguish among M_1 and M_0 . Consequently, his numbers are overestimated by a factor equal to the monetary multiplier.

Aiyagari, Braun, and Eckstein (1998) examine a cash-in-advance economy in the presence of credit goods. There is a continuum of goods which can be acquired in the market in exchange for money or a credit service. Under this second possibility, the price of a good is the money price plus a cost which varies, depending on the good. The higher the inflation rate, the larger the range of goods acquired by credit and, consequently, the higher the money velocity is.⁵ Similar to the present work, their model contemplates that the provision of this money substitutes services by the banking sector requires the employment of production factors, which have been diverted from the real sector. This paper generalizes their findings in many dimensions. It shows that the results depend neither on the specific monetary model taken into consideration nor on the intratemporal elasticity of substitution if the model considers a continuum of goods. In disagreement with Aiyagari et alii, and employing their own calculations, I argue that the distortion effect of inflation on welfare is second-order small, compared to the misallocation effect. Like Lucas's paper, their formulation does not address the distinction among monetary base and demand deposit, as far as the welfare cost of inflation is concern.

The paper is organized as follows. In the Section subsequent to this introduction, the setup of a general version of Sidrauski's model is presented, and in the third Section the welfare characterization and the generality of Bailey's formula are demonstrated. The fourth Section extends the results of the previous Section for the cash-in-advance class of monetary models and discusses the relative merits of the distortion effect *vis-a-vis* the misallocation effect of inflation on welfare. The ensuing Section, applying a version of Sidrauski's model which takes into consideration inside money, clarifies the correct concept of money for this subject - the welfare cost of inflation. The conclusion establishes the main implications of the paper to the measurement of the welfare cost of inflation.

⁵This manner of producing a variable money velocity in cash-in-advance models was introduced by Gillman (1993).

2 The General Model

Usually money can be incorporated into an otherwise standard macroeconomic dynamic model in two ways: as an argument of a shopping time restriction into preferences,⁶ or as an argument of a transaction cost function into the budget constraint.⁷ In order to keep the model exposed here as general as possible, it will be supposed that both possibilities are present. In addition, it is considered that there is another good, along with the traditional good which could be consumed and stocked as capital, called banking service which helps the household in reducing transaction costs, wherever it appears.

Households

The choice problem of the household is the following

$$\max \int_0^\infty e^{-\rho t} u(c_{1t}, s(c_{1t}, m_{1t}, c_{21t})) dt, \quad (1)$$

where u is the instantaneous utility, which is a function of consumption, c_{1t} , and leisure, s . We suppose that labor is inelastically supplied⁸ and, consequently, leisure depends on the time cost of transactions. The lower the consumption of the good, c_{1t} , the higher the quantity of money holding, m_{1t} , and banking services, c_{21} , designated to saving time, the lower the time cost of transaction is. It is supposed that besides this time cost, there is a transaction pecuniary cost attached to consumption. This cost is increasing in the consumption of good, and decreasing in the quantity of money holdings and banking services (respectively m_{2t} and c_{22t}), destined for saving resources cost. Let a_t stand for per capita household's stock of assets, and m_t for per capita household's real quantity of money-retaining. The household maximizes (1), subject to

$$\dot{a}_t = r_t a_t + w_t + \chi_t - c_{1t} - p_t c_{2t} - g(c_{1t}, m_{2t}, c_{22t}) - (\pi_t + r_t) m_t, \quad (2)$$

⁶The specification of preferences as a shopping time restriction was introduced by Saving (1971), although, I will consider an exogenous labor supply. McCallum and Goodfriend (1987) popularized the shopping time formulation in their entry in Palgrave's dictionary. If shopping time does not depend on consumption, we are back to Brock's (1974) perfect foresight formulation of Sidrauski's (1967) model.

⁷See Gray (1984). Feenstra (1986) derived it from some traditional approaches to money microfoundations models and, following a suggestion made by Brock (1974, p. 769), demonstrated the equivalence between this formulation and the money-into-utility approach. For a recent exposition, see Zhang (2000).

⁸In the fourth Section it will be shown that this hypothesis is not essential to the results. It is assumed in this Section because the 2×2 static general equilibrium model is simpler with an exogenous determination of the labor's offer.

given the future path of real assets remuneration, r_t , wages rate, w_t , government transfer, χ_t , relative price of banking services, p_t , and the inflation rate, π_t , where $m_t \equiv \frac{M_t}{P_{1t}}$, $p_t \equiv \frac{P_{2t}}{P_{1t}}$, $a_t \equiv k_t + m_t$,

$$m_t \equiv m_{1t} + m_{2t}, \quad (3)$$

$$c_{2t} \equiv c_{21t} + c_{22t}, \quad (4)$$

g is the transaction-cost function, M_t is the nominal per capita money stock, P_{1t} is the nominal price of the first good, P_{2t} is the nominal price of the banking service, and k_t is the per capita capital stock.

This is a very general model.⁹ For example, if it is supposed that leisure depends only on the quantity of money and if there are no banking sector and transaction costs, we are back to the Sidrauski model. On the other hand, if it is assumed that the instantaneous utility depends only on consumption and that the banking sector does not exist, then we are back to the Feenstra (1986) transaction-cost model of money demand. Finally, if it is supposed that leisure depends only on money and banking services and that there are no transaction costs, the model becomes a simple two-sector model which could rationalize the idea of a banking sector. It is possible to imagine any combination of these three models.

First-Order Conditions

Let λ_t represent the costate variable associated with the restriction (2), which is obviously the shadow price of income. The maximization problem of the household is a standard one.

⁹The standard assumptions are: $u_i > 0$, $l_1 < 0$, $l_i > 0$, $g_1 > 0$, $g_i < 0$ and conditions that assure that utility is strictly concave and household's budget constraint is convex, and, consequently, the equilibrium path exists and is unique (evidently, ruling out monetary bubbles). In particular, g is assumed strictly convex. Additionally, u and g are C^2 functions, such that the path of the variables are differentiable (Oniki, 1973).

The control variables are:¹⁰ c_1 , m_1 , c_{21} , m_2 and c_{22} . It follows the first-order conditions

$$u_1 + u_2 s_1 = \lambda(1 + g_1), \quad (5)$$

$$u_2 s_2 = \lambda(\pi + r), \quad (6)$$

$$u_2 s_3 = \lambda p, \quad (7)$$

$$-g_2 = \pi + r, \quad (8)$$

$$-g_3 = p. \quad (9)$$

For the household state variable (assets), the Euler equation follows

$$\frac{\dot{\lambda}}{\lambda} = \rho - r.$$

Firms

This economy is a two-sector economy. The first sector, applying a linearly homogenous production function which employ capital and labor, produces a good which could be consumed or accumulated as capital. The second sector, applying an equivalent technology, produces a service called banking services, which could be acquired by the household in the market. It is assumed that the factors market clears continuously; factors are perfectly mobile across sectors and are inelastically supplied. Under these conditions, the equilibrium of the supply side of the economy could be represented by the following two supply functions (one for each sector)¹¹

$$y_1 = y_1(p, k) \text{ and } y_2 = y_2(p, k),$$

where y_i is the per capita production of the i -th good.

From the inclination of the possibilities production frontier it is known that¹²

$$y_{11} + p y_{21} = 0, \quad (10)$$

¹⁰The time subscript will be omitted whenever the understanding is clear.

¹¹See Kemp (1969), chapter 1.

¹² $y_{i1} \equiv \frac{\partial y_i}{\partial p} \Big|_k$ and $y_{i2} \equiv \frac{\partial y_i}{\partial k} \Big|_p$.

and from the marginal impact of capital it is known that

$$\frac{\partial}{\partial k}(y_1 + py_2)\Big|_p = y_{12} + py_{22} = f'_1(k_1(p)) = pf'_2(k_2(p)) = r \quad (11)$$

where f_i is the i -th sector product per worker, and k_i is the i -th sector capital per worker ratio.

Government

As it is standard in this literature, it is supposed that the economy works under the monetary regime; the unique role of the government is to print money. For this kind of economy the Friedman rule is satisfied. Although it is an open question¹³ whether, in presence of other imperfections, to inflate the price index is a second-best policy or not, the monetary regime provides a benchmark and an analytical workable solution. Under the monetary regime, the government transference to the public is the *seigniorage* which is equal to the inflationary tax plus the increase in the real quantity of money. That is

$$\chi = \dot{m} + \pi m.$$

Short Run Equilibrium and Dynamics

The market for banking services clears continuously, which means that its relative price (p) adjusts to accomplish this equilibrium. Due to Walras's law, this equilibrium condition, plus the equilibrium in the money market, implies the equilibrium of the goods market. The condition for the equilibrium in the banking services market

$$y_2(p, k) - c_2 = 0, \quad (12)$$

along with equations (3), (4), (5)-(9), determine c_1 , m_1 , c_{21} , m_2 , c_{22} , p , c_2 and π as function of the state variable k , the costate variable λ , and the costate-like variable m . This establishes the momentary equilibrium for this economy.

The dynamic is given by the following equations

¹³See Lucas (2000) Section 4, and the references therein.

$$\dot{k} = y_1(p, k) - c_1 - g(c_1, m_2, c_{22}), \quad (13)$$

$$\dot{\lambda} = \lambda(\rho - f'_1(k_1(p))), \quad (14)$$

$$\dot{m} = m(\sigma - \pi) \quad (15)$$

where $\sigma \equiv \frac{\dot{M}}{M}$.

A very important case, which will be dealt with later, is the situation in which the technology is the same across sectors. If this is true, although from the demand point of view the two goods are distinct, from the supply point of view they are equal. Under this condition, the economy works as if it was an one-sector economy, which means that the relative price of the banking service is constant and that the interest rate is determined by

$$r = f'(k).$$

It follows in this situation, from this last equation and (14), evaluated in the steady-state, that the long-run capital stock is fixed and independent of σ . That is, after an alteration of the growth rate of the nominal quantity of money, the economy will not present any dynamics. The following variables - the control variable, the costate-like variable, and the costate variable - jump, and a new long-run equilibrium is immediately attained.

3 The Impact On Welfare

In this representative agent economy, welfare is equal to the intertemporal utility of the household, expression (1). Let R_t be the nominal interest rate. The following two propositions characterize welfare for this general version of the Sidrauski's model.

Proposition 1 The marginal impact on welfare of an increase in the growth rate of the nominal quantity of money is

$$\frac{dW}{d\sigma} = \int_0^\infty e^{-\rho t} \lambda_t R_t \frac{dm_t}{d\sigma} dt. \quad (16)$$

Proof. See appendix A.1. ■

This expression asserts that with the exception of money, the other choice variables present a social benefit and a social cost, which by the choice mechanism are equal, although welfare theorems are not satisfied for monetary models.¹⁴ In others words, this is a welfare maximizing economy restricted to the fact that the household is consuming less monetary services than the social optimum. That is, a Social Planner who can not avoid inflation, and who can not **induce** the households to increase their money holdings, will have do no better than the market. Consequently, because money has benefit but does not have cost, the amount expressed by (16) remains. Formally,

Proposition 2 If the instantaneous utility is strictly concave and the transaction cost function is strictly convex, the trajectory which satisfies the first-order conditions, the transversality conditions, and the market equilibrium equations is Pareto-Optimum restricted to a given path of the real quantity of money.

Proof. See appendix B.1. ■

In other words, although this economy is not at a Pareto optimum, (16) asserts that **any policy** that increases the present value of money holdings is welfare improving. For instance, an increase of the tax rate on any good, from a initial situation in which taxation is absent, is welfare improving if an only if it increases the present value of the money holdings. Consequently, the result (16) represents a step forward from the dismal position brought by the second best theorem to this model, as far as policy is concern.

It is important to note that there was no supposition about the specific value of σ in deriving the result (16), which means that expression applies to every value for σ , and, consequently, it is a global result.¹⁵ This result states that the marginal impact of σ on welfare is the present value, in units of utilities, of the marginal impact of σ on the money

¹⁴The derivation of this result resembles Samuelson's envelop theorem; however, it is not quite the same. In deriving the envelop theorem for a restricted maximum, the restriction faced by the decision maker is added to the indirect utility function. Differently, in order to derive (16), the restrictions seen by the social planer, which are the physical balance equation for the goods produced by the economy, was added to the indirect welfare function.

¹⁵In deriving (16) no hypothesis was made with respect to the variable σ . That is to say, σ could be any exogenous variable. As an example, if it had been supposed that there was a purchase tax for any good, following the same route which leads us to (16), it would send us to

$$\left. \frac{dW}{d\tau} \right|_{\tau=0} = \int_0^\infty e^{-\rho t} \lambda R \left. \frac{dm}{d\tau} \right|_{\tau=0} dt,$$

demand. The specific adjustment which takes place following an alteration on σ does not matter; the money demand reflects it. Another consequence of (16) is that if the household has a higher initial income she will increase her money demand. In other words, the increase in initial capital necessary to keep constant household's welfare is¹⁶

$$\int_0^\infty e^{-\rho t} \lambda_t R_t \frac{d\bar{m}_t}{d\sigma} dt = \lambda_0 \frac{dk_0}{d\sigma}, \quad (17)$$

where the bar over the money demand remind us that this is the compensate demand.

Let's suppose that the long-run capital stock is not sensitive to the inflation rate.¹⁷ Defining the welfare cost of inflation as the compensate income, recalling that $\rho = r^*$, it follows from the integration of (17) that¹⁸

$$\begin{aligned} \Delta W_{\text{Compensate Income}} &\equiv \int_{-\rho}^{\sigma} \frac{dC. \text{Income}_0(\sigma)}{d\sigma} d\sigma = \int_{-\rho}^{\sigma} r^* \frac{dk_0}{d\sigma}(\sigma) d\sigma \\ &= \int_{-\rho}^{\sigma} R^* \frac{d\bar{m}^*}{d\sigma'} d\sigma' = - \int_{\bar{m}^*(\sigma)}^{\bar{m}^*(-\rho)} R^*(\bar{m}) d\bar{m}. \end{aligned} \quad (18)$$

For a very general class of monetary models, the area under the inverse compensate money demand function is the accurate **general equilibrium measure** of the impact of inflation on welfare. Said differently:

“This conclusion, that the area under the observed demand curve for real cash balances during an inflation measures the welfare costs of the reduction of these balances, applies **regardless** of the particular manner in which these costs affect real income and leisure.” (Bailey, (1956), pg.102, emphasis added.)

The next Section shows that the validity of (16) and (18) are not an artifact of the Sidrauski model or the transaction cost version of it.

in which τ is the tax rate. The important distinction is that this result would apply in the neighborhood of the tax rate close to zero; in contrast, due to the particular role played by the parameter σ in monetary models - generally, a parameter displaces some first-order conditions and that is not the case regarding to σ - (16) is a global result.

¹⁶See Appendix A.1. Note that (17) is, like (16), a global result.

¹⁷As it was seen, it is necessary to assume that technology is the same among sectors.

¹⁸The ‘*’ indicates that the results refer to a steady state capital stock that does not change with inflation.

4 A Cash-in-Advance Economy

In addition to the Sidrauski family of monetary models the other workhorse of applied monetary theory is the family of the cash-in-advance models. The aim of this Section is to demonstrate that the results which were derived for the Sidrauski-type models are valid to this family of monetary models. The same route will be followed; for a very general cash-in-advance model, which could encompass many models as a particular case, (16) and (18) will be established.

The drawback of the standard¹⁹ cash-in-advance model is the constancy in income velocity. The manner which has been suggested to cope with this limitation is to add goods that can be purchased by credit.²⁰ As put forth by Gillman (1993), it is possible to consider a continuum of goods, which, from the preference point of view possesses symmetric roles, although not from the transaction technology point of view. Under this formulation, every good can be purchased by money or credit. The distinction is that there is a credit cost attached to each good which varies across goods, in such a way that as inflation increases, the range of goods which are credit goods increases. If it is considered that these credit services are offered by a sector of the economy which employs production factors in order to produce it, we are in the Aiyagari, Braun, and Eckstein (1998) or English (1999), framework.

The model that will be studied in this Section is a generalization of Aiyagari's *et alii* model²¹ in one direction; the aggregator function, which defines the consumption good and the investment good, presents elasticity of substitution across types of goods larger than zero. There are two main reasons for this choice. Firstly, it is intended to work in a more general set up, which can deliver other models as a particular case. Secondly, the situation in which the elasticity across types of goods is higher than zero produces another impact of inflation on welfare. Due to the symmetric role played by the goods in preference, the household prefers to smooth consumption across types. Notwithstanding this, among the goods acquired as credit goods, the relative price - the credit cost relative to the nominal interest rate - varies in such a way that following an increase in inflation rate, the variability of consumption across types increases. This is a relatively rich description of a monetary economy under certainty.

¹⁹For example, Lucas (1981a).

²⁰Lucas and Stokey (1983).

²¹The results of this section apply to English's (1999) version of the cash-in-advance model with many goods.

Following an increase in inflation, the range of cash goods decreases, the consumption profile of the household twists, the banking sector absorbs production factors to offer transaction services, and the accumulation of capital is hindered. However, it will be shown that (16) represents the marginal impact on welfare of inflation. Moreover, if it is supposed that capital accumulation is not affected by inflation, Bailey's formula is again valid.

4.1 The Model

There is a continuum of goods index by $z \in [0, 1]$. They are identical goods from the supply point of view, which means that the producer price P_t is the same, regardless of the type.²² There is another sector in this economy, the banking sector, which produces a service. Each good could be acquired as cash good or credit good. In the first case, the household pays P_t , but has to have it as cash, which means that the cost it faces is $P_t(1 + R_t)$. When buying a good as credit good, the household pays P_t to the good's producer plus the intermediation services cost. Following Aiyagari et alii, it is supposed that to acquire a unit of good of any quality as credit good, it is necessary to buy $R(z)$ units of banking services, which cost $pR(z)$ in units of goods. Consequently, the effective cost of a credit good to the household is $P_t(1 + pR(z))$. It is supposed that the production function for goods and transaction services are the same, which means that it is possible to normalize $p = 1$. The total per capita production of goods and services is $f(k_t, n_t)$, where n_t is the per capita supply of labor services. Moreover, the transaction services cost function is increasing in the index z and $R(0) = 0$. At any moment there is a cut-off index, z_t , such that any good whose index is lower than the cut-off is bought as credit good, and the others are bought as cash.

Household Choice

The household solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) \quad (19)$$

where

$$c_t = \left(\int_0^1 c_t^{\frac{\theta-1}{\theta}}(z) dz \right)^{\frac{\theta}{\theta-1}}$$

is an aggregator function that defines the unit of consumption.

²²This subsection follows closely Aiyagari et alii.

The household faces two sorts of restrictions. One is the cash-in-advance and the other is the budget constraint. Before going to the good market, it is possible to go to the credit market, in order to take cash. This operation is without cost. Let M_t , B_t , and X_t be, respectively, the nominal quantity of money and bonds in the household portfolio, and the nominal value of government transfer. The cash-in-advance restriction is

$$\frac{M_t + X_t}{P_t} + \frac{B_t}{P_t} - \frac{B_{t+1}}{P_t(1 + R_t)} \geq \frac{1}{P_t} \int_{z_t}^1 P_t(z)(c_t(z) + i_t(z))dz. \quad (20)$$

The left side of (20) is the amount of cash carried for consumption before going to the goods market in the instant t , and the right side is the nominal cost of cash goods. The budget constraint is

$$\frac{M_t + X_t}{P_t} + \frac{B_t}{P_t} + w_t n_t + r_t k_t \geq \frac{1}{P_t} \int_0^1 P_t(z)(c_t(z) + i_t(z))dz + \frac{M_{t+1}}{P_t} + \frac{B_{t+1}}{P_t(1 + R_t)}. \quad (21)$$

The movement equation for capital is

$$k_{t+1} = i_t + (1 - \delta)k_t, \quad (22)$$

where i_t is an aggregator function that defines the investment good

$$i_t = \left(\int_0^1 i_t^{\frac{\theta-1}{\theta}}(z) dz \right)^{\frac{\theta}{\theta-1}}.$$

Taking the limit $\theta \rightarrow 0$ this model delivers the Aiyagari et alii model; the limit $\theta \rightarrow 1$ reproduces Gillman's model if an economy without capital is considered. If the cut-off index, z_t , is fixed and if there are neither banking services nor transaction services, the model reproduces Lucas and Stokey's (1983) economy under certainty, and if there are no credit goods, the model generates Stockman's (1981) model. Additionally, if capital is a credit good without transaction cost, Lucas's (1981a) model under certainty is obtained.

First-Order Conditions

For this constant-substitution-elasticity aggregator, it is known that

$$\left(\frac{c_t(z)}{c_t}\right)^{-\frac{1}{\theta}} = \left(\frac{i_t(z)}{i_t}\right)^{-\frac{1}{\theta}} = (1 + R(z)) \frac{P_t}{Q_t} \text{ if } z \leq z_t \quad (23)$$

$$\text{and } \left(\frac{c_t(z)}{c_t}\right)^{-\frac{1}{\theta}} = \left(\frac{i_t(z)}{i_t}\right)^{-\frac{1}{\theta}} = (1 + R_t) \frac{P_t}{Q_t} \text{ if } z > z_t, \quad (24)$$

if the household faces the price $(1 + R(z)) P_t$ when $z \leq z_t$, and faces the effective price $(1 + R_t) P_t$ when $z > z_t$, where

$$Q_t \equiv P_t(1 + \tau_t) \equiv P_t \left[\int_0^{z_t} (1 + R(z))^{1-\theta} dz + (1 - z_t)(1 + R_t)^{1-\theta} \right]^{\frac{1}{1-\theta}} \quad (25)$$

is the effective price index faced by the household.

Let $\beta^t \lambda_t \mu_t$, $\beta^t \lambda_t$, and $\beta^t \lambda_t q_t$ be respectively the Lagrange multipliers of (20), (21), and (22). Recalling that $P_t(z) = (1 + R(z)) \frac{P_t}{Q_t}$ if $z \leq z_t$ and that $P_t(z) = P_t$ if $z > z_t$, it follows that the first-order conditions for the flows variables, consumption and investment, are

$$\begin{aligned} u_1(c_t, 1 - n_t) c_t^{\frac{1}{\theta}} c_t^{-\frac{1}{\theta}}(z) &= \lambda_t(1 + \mu_t) \\ \text{and } q_t i_t^{\frac{1}{\theta}} i_t^{-\frac{1}{\theta}}(z) &= 1 + \mu_t \text{ if } z > z_t; \end{aligned} \quad (26)$$

$$\begin{aligned} u_1(c_t, 1 - n_t) c_t^{\frac{1}{\theta}} c_t^{-\frac{1}{\theta}}(z) &= \lambda_t(1 + R(z)) \\ \text{and } q_t i_t^{\frac{1}{\theta}} i_t^{-\frac{1}{\theta}}(z) &= 1 + R(z) \text{ if } z \leq z_t. \end{aligned} \quad (27)$$

The first-order conditions for the labor supply and the cut-off index are

$$u_2(c_t, 1 - n_t) = \lambda_t w_t$$

and

$$1 + \mu_t = 1 + R(z_t).$$

This last condition states that the relative price of money in units of bonds is equal to the credit cost of the cut-off good. This relative price should be equal to the nominal interest rate in order to keep the Budget restriction bounded; otherwise it would be possible to gain money selling (or buying) cash the z_t good, and buying (or selling) it as credit good. At each instant the cut-off good is determined with the aim of meeting this non-arbitrage condition.

That is

$$\mu_t = R_t = R(z_t). \quad (28)$$

As Gillman (1993) stressed, (28) is a Baumol-type condition which equates the marginal cost of hold money with the marginal transaction cost.

After substituting (23) and (24) into (26) and (27), recalling (25) and (28), it follows that

$$u_1(c_t, 1 - n_t) = \lambda_t(1 + \tau_t) \text{ and } q_t = 1 + \tau_t. \quad (29)$$

The Euler equations for capital and bonds are respectively

$$\lambda_t(1 + \tau_t) = \beta \lambda_{t+1}(1 + \tau_{t+1})(1 - \delta + \frac{r_{t+1}}{1 + \tau_{t+1}}) \quad (30)$$

and,

$$\lambda_t = \beta \lambda_{t+1}(1 + R_{t+1}) \frac{P_t}{P_{t+1}}.$$

It is apparent from (30), after substituting (29), that the cash-in-advance restriction on investment acts as a distortion taxation on capital. As it will be seen, this is not true from the social point of view.

4.2 Impact on Welfare

Proposition 3 The marginal impact on welfare of an increase in the growth rate of the nominal quantity of money is

$$\frac{dW}{d\sigma} = \sum_{t=0}^{\infty} \beta^t \lambda_t R_t \frac{dm_t}{d\sigma}. \quad (31)$$

Proof. See appendix A.2. ■

Equation (31) is equivalent to (16). Although from looking at the Euler equation it is apparent that there is a distortion tax on capital accumulation, this is not true from the social point of view. Notwithstanding the fact that the cash-in-advance restriction applies to the investment decision, (31) shows us that the capital-stock path is at an extremum of the welfare function. It is straightforward to recalculate (31) for the case in which there is an explicit distorted tax on capital income, whose proceeds is rebounded to the household

in a lump sum fashion. One will get

$$\frac{dW}{d\sigma} = \sum_{t=0}^{\infty} \beta^t \lambda_t \left(R_t \frac{dm_t}{d\sigma} + \tau_{Kt} r_t \frac{dk_t}{d\sigma} \right),$$

where τ_K is the tax rate on capital income. When there is an explicit taxation on capital, variations on the capital accumulation path has a first-order impact on welfare.

Consequently, the conclusion that the monetary models are Pareto-Optimum restricted applies to the cash-in-advance class of models. Formally,

Proposition 4 The solution for the Central-Planner problem, restricted to the cash-in-advance restriction and to a given path for the real quantity of money, replicates the market solution.

Proof. See appendix B.2. ■

Continuing along the same path that was taken in the first part of the paper, let's suppose that the economy presents a long-run capital stock that does not vary with σ . Integrating (31), considering the compensate income, Bailey's formula follows²³

$$\Delta W_{\text{Compensate Income}} = - \int_{\bar{m}^*(\sigma)}^{\bar{m}^*(-\rho)} R(\bar{m}) d\bar{m}.$$

For this economy, Bailey's formula is the measure, in units of assets, of the impact on welfare of inflation. The area under the inverse compensate money demand function takes into consideration firstly the inflow of production factors into the banking sector and the reduction of labor supply,²⁴ which results in the decrease of the average consumption level, and, secondly, the increase in the variability of consumption across types of consumption goods.

Discussion

In this model, inflation has two impacts on welfare: Firstly, the allocation effect - the increase in inflation diverts resources from the goods sector towards the banking sector, affects the

²³See Appendix A.2. Aiyagari et alii (1998) derived this result for their economy. They did not realize that the area should be taken over the compensate demand. In this paper I consider the Hicksian demand function; they considered the constant-real-income demand function.

²⁴In the models of the first part of this paper, it was supposed that the labor supply was inelastic.

labor supply, and the consumption profile across types of goods. Secondly, the distortion effect or intertemporal allocative effect - since the cash-in-advance restriction applies to investment good, inflation increases the shadow price of capital and, consequently, reduces steady-state capital stock. A natural question is how do these two effects compare. A definitive answer is possible only from computation analysis. Notwithstanding, it is possible to accomplish an assessment of its relative merits with the information that we have so far. As was seen, the interpretation of (31) is that this model is Pareto-Optimum restricted. For any value of σ , money holdings is the unique variable which is not optimum-chosen from the social point of view; the other variables, including capital stock, are at an extremum of the Welfare Function. It follows from this argument that the misallocation effect has a first-order impact on welfare, and the distortion effect is second-order small. Aiyagari's et alii provides calculations of both effects for this cash-in-advance model when $\theta = 0$. In their figure 6 (p. 1298), the total welfare cost with transition is reported, and in their figure 5 (p. 1295), the misallocation cost is reported. Subtracting the last from the former, the result is that the distortion effect is almost nil, as it would be expected in the face of (31).²⁵ In other words, it is a consequence of (31) that the distortion effect, taken into consideration the transitory dynamic, would be considerable **only if the impact of inflation on capital accumulation have had produced sizable movements on the money demand**, which is not the case under standard calibration specification.

5 A Model with Inside Money

As it was seen in Section three, the first implication of the welfare characterization of monetary-applied models brought about in this paper is the following: Abstracting from impacts of inflation under long-run capital, the area under the compensate inverse money demand function is the accurate measure of the reduction on welfare caused by perfectly

²⁵This observation contrast with Aiyagari's et alii discussion. According to them:

“The second result is that at low to moderate inflation rates, the inflation distortion tax component, which is the difference between the total welfare cost and the misallocation component, is roughly from two to three times the misallocation component.” (pg. 1298)

Their result rests on their across stationary-state welfare comparisons. They did not realize that the inclusion of the transitory dynamics not only reduces the welfare cost, but, quantitatively, practically eliminates the distortion component!

predicted inflation. This conclusion is quite general and does not depend on the specific role played by money in the economy nor on the specific kind of adjustment faced by the real sector to avoid or to help the public to cope with inflation. Moreover, if this measure is not exact, due to failure of long-run capital's neutrality, the discrepancy between this measure and the actual is second-order small, which is the third implication of the welfare characterization. Consequently, the next stage is to determine what monetary aggregate should be employed to perform the welfare cost calculation. What is money? Whenever the researcher is studying the short-run equilibrium of the economy, money is the asset which possesses the property of liquidity. Money is usually cash out of the banking sector plus demand deposits. But, that is not what is meant by money in this context. With respect to this issue - welfare cost of inflation - money is that good which has benefit **but does not have social cost.**^{26,27}

When inflation increases, the public demand for demand deposits decreases, which could be considered a welfare cost of inflation. However, because this service - demand deposit - requires capital and work force to be supplied, the reduction in the public demand for demand deposit is not a cost, from the social point of view. What occurs is that the increase of inflation decreases the demand-deposit demand, but it increases the demand for the other bank services in such a way that the demand for an aggregated bundle of banking services increases. The variant of the second Section model sketched below argues that the monetary base is the relevant concept of money for evaluating the welfare cost of the perfect foreseen inflation.

Household

There are three liquidity instruments: cash, m_{1t} , demand deposits, m_{2t} , and another banking

²⁶This concept of money applies to Friedman's rule. The asset whose consumption should be pushed to satiation is the monetary base.

²⁷Differently, Lucas (1981b) pg. 44, defines money, as far as the welfare impact of inflation is concerned, as any

“noninterest-bearing assets or to assets the interest on which is restricted to below-market rates.”

In the same Section he offers a discussion of the money concept and its role as a liquidity instrument. The point here is that the precise way that money takes place in the economy - if it provides liquidity or if there are restrictions and regulation in its usage - is not the heart of the question, which is that money has social value and does not have social cost.

service, c_{2t} . The household solves

$$\max \int_0^\infty e^{-\rho t} u(c_{1t}, s(m_{1t}, m_{2t}, c_{2t})) dt, \quad (32)$$

subject to

$$\dot{a}_t = r_t a_t + w_t + \chi_{H,t} + \varkappa_t - c_{1t} - p_t c_{2t} - p_t^d m_{2t} - (\pi_t + r_t)(m_{1t} + m_{2t}), \quad (33)$$

where²⁸ $a \equiv k_h + m_1 + m_2$, k_h is household's physical capital stock, $\chi_{H,t}$ is the Government transfers to the household, \varkappa_t is the bank's profits, and p_t^d is the demand deposit price. For simplicity, the other banking services are treated as flow of services and not as assets. Because of the possibility of very low inflation rates, the banks charge a fee to held demand deposits.²⁹ It is possible, if inflation is sufficiently high, that this price could be negative. However, usually the banking system is regulated, such that

$$p^d \geq 0. \quad (34)$$

The first-order conditions for this standard problem is

$$u_1 = \lambda, \quad (35)$$

$$u_2 s_1 = \lambda(\pi + r), \quad (36)$$

$$u_2 s_2 = \lambda(\pi + r + p^d), \quad (37)$$

$$u_2 s_3 = \lambda p, \quad (38)$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - r. \quad (39)$$

The Banks

This is a two-sector economy. The first sector produces a good, which can be consumed and accumulated as capital. The second sector, banks, in this Section are multiproduct firms.

²⁸Nothing would change if this model had been built up as general as the model in the section two.

²⁹I am assuming that the household demands demand-deposit because she can benefit from the services provide by the banking institution to the demand-deposit holder. As usual in production theory, I assume that the flow of services is proportional to the stock. Examples of demand-deposit services are check redemption, the payment of bills, and the supply of automatic cashier on the streets.

They employ capital and work force to produce a service (called banking services, which help the household in saving transaction time), and to produce another liquidity service, named demand deposit. As usual, it is supposed that the demand deposits are denominated in nominal units. The household, to open a checking account, deposits goods in the bank. The bank creates a deposit denominated in nominal units and rents these goods to the firms. Consequently, the income of the banking in offering this services is the price that it could charge plus the nominal interest rate. Therefore, the *per capita* profit function for the banks, in units of goods, are

$$\pi = pc_2 + (p^d + (\pi + r)(1 - \zeta))m_2 - (rk_2 + w)l_2 + \chi_B, \quad (40)$$

where ζ is the reserves requirement ratio, k_2 is the capital-labor ratio in the banking sector, l_2 is the ratio of the work force employed by the banking sector, and χ_B is the Govern's transfer to the Banks.

In this set up, the demand deposit has triple significance. First, it is a nominal asset which belongs to household's portfolio, as it is clear from (33). Second, it is a part of the economy's physical capital, whose owners are the banks. Finally, it is a service which is acquired by the household; the household, after depositing goods in the demand deposit, is entitled to use the flow m_2 of services provided by the banks. In order to offer this service, the bank employs production factors, as is clear from (40), and receives $p^d + (\pi + r)(1 - \zeta)$ per unit of service. This 'price' has three components, each one related to one of the demand deposit's significance. First, because it is a nominal asset, the inflationary tax is an income appropriated by the bank. Second, the firms pay rent for using bank's capital stock, $(1 - \zeta)m_2$. Finally, because it is a service, the bank can charge a fee.

The banks maximize (40), subject to the technological restriction³⁰

$$y_2 = l_2 f_2(k_2) = g(c_2, m_2), \quad (41)$$

where $f_i \equiv \frac{F_i(L_i, K_i)}{L_i}$ is i -th sector's per worker output; L , L_i , and K_i , are, respectively, the total labor supply, labor's services allocated to the i -th sector, and capital's services allocated to the i -th sector.

Restriction (41) states that the *per capita* production of this industry can be distributed

³⁰This modeling of a multiproduct firm was taken from Drazen (1979).

across the two products according to the transformation function g . This function is concave and linearly homogeneous. Let q be the Lagranger multiplier for (41). The first-order conditions for the maximization problem for the banks are as follows

$$p = qg_1, \quad (42)$$

$$p^d + (\pi + r)(1 - \zeta) = qg_2, \quad (43)$$

$$r = qf'_2(k_2), \quad (44)$$

$$w = q(f_2 - k_2 f'_2(k_2)). \quad (45)$$

Due to the homogeneity of g , it follows from (42) and (43) that

$$pc_2 + (p^d + (\pi + r)(1 - \zeta))m_2 = qy_2, \quad (46)$$

which means that the total **per capita** production of the Banks, evaluated in units of goods, is equal to the production of services, priced at p , and the production of demand deposits, priced at $p^d + (\pi + r)(1 - \zeta)$. The price q is the price, in units of goods, of a an optimum bundle of transaction services and demand deposits. This is the relevant price for the allocation decision for the production factors.³¹ At each instant the price q determines the relative rentability across the sectors, and, accordingly, the allocation of factors between the real sector and the banking sector.³² Consequently, the sector's offers function can be written as follows

$$y_1(q, k) \text{ and } y_2(q, k).$$

Similar to the other Sections, proprieties (10) and (11) are satisfied. Given an amount of banking output, y_2 , the relative price between services and demand deposits determines at which point of the transformation function, g , the banking sector will be positioned. On the other hand, equation (46) could be seen as an equilibrium equation for the banking sector. Totally differentiating (46) after substituting

$$dy_2 = q^{-1}(pdc_2 + (p^d + (\pi + r)(1 - \zeta))dm_2),$$

³¹From (44) and (45) it is possible to verify it directly.

³²It is apparent that this economy does not satisfy Friedman's rule for demand deposit. If inflation decrease, to offer this service the banks will charge the fee p^d , in order to pay for the cost of this provision. See footnote 26.

it follows that

$$y_2 dq = c_2 dp + m_2 d(p^d + (\pi + r)(1 - \zeta)). \quad (47)$$

This last result will be useful later.

General Equilibrium and Welfare

Because the transformation frontier for the Banks is linearly homogeneous, the payment of factor by its marginal productivity is equal to the production of liquidity services - $pc_2 + (p^d + (\pi + r)(1 - \zeta))m_2$. Consequently, the bank's profit is government's transfer - χ_B . After substituting the liquidity services equilibrium equation (46), remembering that the per capita non-banking-sector income - $rk_h + w$ - is equal to the per capita output - $y_1 + qy_2$ - net of bank's capital income - $r(1 - \zeta)m_2$ - and that the total government transfer is equal to the seigniorage of the monetary base - $\dot{m}_1 + \zeta\dot{m}_2 + \pi(m_1 + \zeta m_2)$ -, the good's market equilibrium equation follows from (33). One can get

$$\dot{k} \equiv \frac{d}{dt}(k_h + (1 - \zeta)m_2) = y_1(q, k) - c_1. \quad (48)$$

It is possible now to evaluate the impact of inflation on welfare.

Proposition 5 The marginal impact on welfare of an increase in the growth rate of the nominal quantity of money is

$$\frac{dW}{d\sigma} = \int_0^\infty e^{-\rho t} \lambda(\pi + r) \frac{d(m_1 + \zeta m_2)}{d\sigma} dt. \quad (49)$$

Proof. See appendix A.3. ■

Defining $b = m_1 + \zeta m_2$, in which b stands for the monetary base, it follows from (49) that this economy solves for trajectory which is Pareto-Optimum restricted to a path of the monetary base. The following proposition establishes this.

Proposition 6 The Central-Planner's solution, restricted to a given path for the real quantity of the monetary base, reproduces the market solution.

Proof. See appendix B.3. ■

Again, if the capital intensity across sectors is the same, it is possible to integrate (49) to get

$$\Delta W_{\text{Compensate Income}} = \int_{-\rho}^{\sigma} R^* \frac{d\bar{b}^*}{d\sigma'} d\sigma' = - \int_{\bar{b}^*(\sigma)}^{\bar{b}^*(-\rho)} R^*(\bar{b}) d\bar{b}. \quad (50)$$

The results (49), (50), and proposition 6, are valid if (34) is not binding; if it is, the general equilibrium solution of the model will be changed. Particularly, the demand for monetary base, for demand deposit, and for the other transactions-saving services, will be displaced, and, consequently, welfare will be affected by (34). Notwithstanding, the impact of the regulation on welfare is ambiguous. Because welfare theorems are not satisfied for monetary models, the impact on welfare of an additional restriction is not clear, which is a standard second-best result. For this specific institutional restriction, the source of the ambiguity is that on the one hand, (34) reduces welfare because it induces a misallocation of factors towards transaction-saving services and out of demand deposits,³³ on the other hand, it stimulates the demand for currency, which improves welfare. Once we acknowledge that this economy is Pareto-Optimum restricted to a path of the monetary base, it follows that there is no ambiguity if (34) is marginally binding: the increase in welfare due to the increase of the monetary base, a first-order effect, supplants the misallocation of factors towards the provision of transaction-saving services.

6 Conclusion

This paper offers a characterization of welfare property of the applied monetary-models. It shows that these monetary models imply trajectories which are Pareto-Optimum restricted to a given path of the real quantity of money. As a corollary, three implications follow. Firstly, it has been shown that the use of Bailey's formula to evaluate the impact of inflation on welfare is indeed exact for many monetary models, among others the standard Sidrauski's model, the transaction version of the Sidrauski's model, and the cash-in-advance family of models, if the compensate inverse money demand is considered. In particular, the result applies if the existence of a banking sector that provides services which are substitutes for money is taken into consideration. Although the banking sector helps the public to cope with inflation, it extracts production factors which have a positive social value in the good market. Notwithstanding these effects, the measure of the impact on welfare of inflation is the usual one - the area under the compensate inverse demand curve for money. That does not mean that the increase of the banking sector is without consequence. Due to the general equilibrium nature of the problem, if by any reason the banking share in the product had not

³³Competition among banks will increase the price of the transaction-saving services.

been increased, the steady-state money demand would be different. The point here is that all these general equilibrium effects³⁴ that follows from an increase in the inflation rate have the very same analytical expression for the impact on welfare of an increase in the inflation rate, which is exactly expressed by Bailey's formula. Therefore, when one calculates the welfare impact of inflation applying Bailey's formula, the researcher has already taken into consideration the fact that the banking sector has taken real resources from the other sector to provide banking services to the public. And this result is robust whether a Sidrauski-type model or a cash-in-advance model is taken into consideration.

Secondly, it has been argued that the relevant demand function for evaluating the impact of inflation on welfare is the narrow monetary aggregate, the monetary base. This observation follows from the fact that the demand deposit is a service provided by the banking sector, and consequently, requires the employment of production factors to be offered. Consequently, abstracting from the impact of inflation into capital accumulation³⁵ or in long-run growth rate,³⁶ the general equilibrium measurement of the effect of inflation on welfare is the area under the inverse monetary-base demand. This result moderates, for example, Lucas's (2000) estimation. In addition, the works which calculate the welfare effect of inflation, calibrating a general equilibrium model in order to match the observable M_1 demand, overstate the cost.

The third result that follows from the welfare characterization, is that the distortion effect of inflation under capital accumulation in welfare is second-order small. This conclusion, jointly with the conclusion in the previous paragraph, supports the view that the welfare cost of perfectly predicted inflation for very low inflation rates, as has been the case for the developed economies in the last twenty years, is as low as Fischer's (1981) calculations.³⁷

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³⁴The increase in the share of the banking sector, the reduction in the consumption level, the reduction in the quantity of money, and the elevation of the consumption of banking services.

³⁵As, for example in Aiyagari et alii (1998).

³⁶Dotsey and Ireland (1996) investigate the welfare cost of inflation in an endogenous growth framework.

³⁷The reader should remind that (34) is not binding if inflation is very low.

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A Appendix³⁸

A.1 Proposition 1

Proof. From (1), it follows that

$$\begin{aligned}\frac{dW}{d\sigma} &= \int_0^\infty e^{-\rho t} \frac{d}{d\sigma} u(c_1, s(c_1, m_1, c_{21})) dt \\ &= \int_0^\infty e^{-\rho t} \left[(u_1 + u_2 s_1) \frac{dc_1}{d\sigma} + u_2 s_2 \frac{dm_1}{d\sigma} + u_2 s_3 \frac{dc_{21}}{d\sigma} \right] dt.\end{aligned}\quad (51)$$

Substituting in this last equation the first-order conditions (5)-(7), it follows that

$$\frac{dW}{d\sigma} = \int_0^\infty e^{-\rho t} \lambda \left[(1 + g_1) \frac{dc_1}{d\sigma} + (\pi + r) \frac{dm_1}{d\sigma} + p \frac{dc_{21}}{d\sigma} \right] dt.$$

From the equilibrium in the market for goods, equation (13), it is known that

$$\int_0^\infty e^{-\rho t} \lambda \frac{d}{d\sigma} \left(y_1(p, k) - c_1 - g(c_1, m_2, c_{22}) - \dot{k} \right) dt = 0$$

and for the banking services market, equation (12), it follows that

$$\int_0^\infty e^{-\rho t} \lambda p \frac{d}{d\sigma} (y_2(p, k) - c_2) dt = 0,$$

which could respectively be written as

$$\int_0^\infty e^{-\rho t} \lambda \left(y_{11} \frac{dp}{d\sigma} + y_{12} \frac{dk}{d\sigma} - (1 + g_1) \frac{dc_1}{d\sigma} - g_2 \frac{dm_2}{d\sigma} - g_3 \frac{dc_{22}}{d\sigma} - \frac{d\dot{k}}{d\sigma} \right) dt = 0, \quad (52)$$

and

$$\int_0^\infty e^{-\rho t} \lambda p \left(y_{21} \frac{dp}{d\sigma} + y_{22} \frac{dk}{d\sigma} - \frac{dc_2}{d\sigma} \right) dt = 0. \quad (53)$$

³⁸Oniki (1974) showed that the solutions of continuous-time dynamic-optimization problems are differentiable.

Integrating by parts the last term in (52), recalling that capital is bounded and the transversality conditions, it follows that

$$\int_0^\infty e^{-\rho t} \lambda \frac{d}{dt} \frac{dk}{d\sigma} dt = -\lambda_0 \frac{dk_0}{d\sigma} - \int_0^\infty e^{-\rho t} \lambda \left(-\rho + \frac{\dot{\lambda}}{\lambda}\right) \frac{dk}{d\sigma} dt. \quad (54)$$

Substituting (54) in (52), adding the result and (53) to (51) it is left

$$\begin{aligned} \frac{dW}{d\sigma} = & \int_0^\infty e^{-\rho t} \lambda \left\{ \left[(1 + g_1) \frac{dc_1}{d\sigma} + (\pi + r) \frac{dm_1}{d\sigma} + p \frac{dc_{21}}{d\sigma} \right] \right. \\ & + \left(y_{11} \frac{dp}{d\sigma} + y_{12} \frac{dk}{d\sigma} - (1 + g_1) \frac{dc_1}{d\sigma} - g_2 \frac{dm_2}{d\sigma} - g_3 \frac{dc_{22}}{d\sigma} + \left(-\rho + \frac{\dot{\lambda}}{\lambda}\right) \frac{dk}{d\sigma} \right) \\ & \left. + p \left(y_{21} \frac{dp}{d\sigma} + y_{22} \frac{dk}{d\sigma} - \frac{dc_2}{d\sigma} \right) \right\} dt - \lambda_0 \frac{dk_0}{d\sigma}. \end{aligned}$$

After recalling (8), (9), (10), (11), and (14), every term which is not multiplied by $\frac{dm_i}{d\sigma}$ cancels out. Considering the primal approach, it remains (16); considering the dual approach we get (17). ■

A.2 Proposition 3

Proof. From (19), after substituting the first-order conditions (26) and (27), recalling (23) and (24), it follows that

$$\frac{dW}{d\sigma} = \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_0^{z_t} (1 + R(z)) \frac{dc_t(z)}{d\sigma} dz + \int_{z_t}^1 (1 + R_t) \frac{dc_t(z)}{d\sigma} dz - w_t \frac{dn_t}{d\sigma} \right]. \quad (55)$$

The material balance equation for this economy is

$$f(k_t, n_t) - \int_0^{z_t} (1 + R(z))(c_t(z) + i_t(z)) dz - \int_{z_t}^1 (c_t(z) + i_t(z)) dz = 0,$$

which means that

$$\begin{aligned}
0 = & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t \frac{dk_t}{d\sigma} + w_t \frac{dn_t}{d\sigma} - \int_0^{z_t} (1 + R(z)) \left(\frac{dc_t(z)}{d\sigma} + \frac{di_t(z)}{d\sigma} \right) dz \right] \\
& - \sum_{t=0}^{\infty} \beta^t \lambda_t \left[R(z_t) (c_t(z_t) + i_t(z_t)) \frac{dz_t}{d\sigma} + \int_{z_t}^1 \left(\frac{dc_t(z)}{d\sigma} + \frac{di_t(z)}{d\sigma} \right) dz \right].
\end{aligned} \tag{56}$$

Adding (56) to (55), it follows that

$$\begin{aligned}
\frac{dW}{d\sigma} = & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_{z_t}^1 R_t \frac{dc_t(z)}{d\sigma} dz - R(z_t) c_t(z_t) \frac{dz_t}{d\sigma} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t \frac{dk_t}{d\sigma} - \int_0^{z_t} (1 + R(z)) \frac{di_t(z)}{d\sigma} dz - \int_{z_t}^1 \frac{di_t(z)}{d\sigma} dz - R(z_t) i_t(z_t) \frac{dz_t}{d\sigma} \right].
\end{aligned} \tag{57}$$

From the first-order condition for the investment, it follows that

$$(1 + \tau_t) i_t = \int_0^{z_t} (1 + R(z)) i_t(z) dz + (1 + R_t) \int_{z_t}^1 i_t(z) dz,$$

which means that

$$\begin{aligned}
0 = & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_0^{z_t} (1 + R(z)) \frac{di_t(z)}{d\sigma} dz + (1 + R_t) \int_{z_t}^1 \frac{di_t(z)}{d\sigma} dz \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\frac{dR_t}{d\sigma} \int_{z_t}^1 i_t(z) dz - i_t \frac{d(1 + \tau_t)}{d\sigma} - (1 + \tau_t) \frac{di_t}{d\sigma} \right].
\end{aligned} \tag{58}$$

Adding (58) to (57), recalling that

$$\frac{dR_t}{d\sigma} \int_{z_t}^1 i_t(z) dz - i_t \frac{d(1 + \tau_t)}{d\sigma} = 0,$$

it follows that

$$\begin{aligned}
\frac{dW}{d\sigma} = & \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\int_{z_t}^1 R_t \frac{d}{d\sigma} (c_t(z) + i_t(z)) dz - R(z_t) (c_t(z_t) + i_t(z_t)) \frac{dz_t}{d\sigma} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t \frac{dk_t}{d\sigma} - (1 + \tau_t) \frac{di_t}{d\sigma} \right].
\end{aligned} \tag{59}$$

From the capital accumulation equation it is possible to rewrite the second line in (59) as

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t \lambda_t \left[r_t \frac{dk_t}{d\sigma} - (1 + \tau_t) \left(\frac{dk_{t+1}}{d\sigma} - (1 - \delta) \frac{dk_t}{d\sigma} \right) \right] \\
&= \sum_{t=1}^{\infty} \left[\beta^t \lambda_t r_t + \beta^t \lambda_t (1 + \tau_t)(1 - \delta) - \beta^{t-1} \lambda_{t-1} (1 + \tau_{t-1}) \right] \frac{dk_t}{d\sigma} + \lambda_0 [r_0 + (1 + \tau_0)(1 - \delta)] \frac{dk_0}{d\sigma} \\
&= \lambda_0 [r_0 + (1 + \tau_0)(1 - \delta)] \frac{dk_0}{d\sigma}, \tag{60}
\end{aligned}$$

in which the second equality follows from the first-order condition for capital accumulation, equation (30), the transversality condition, and because capital is bounded. Substituting (60) into (59), it remains

$$\begin{aligned}
\frac{dW}{d\sigma} &= \sum_{t=0}^{\infty} \beta^t \lambda_t R_t \frac{d}{d\sigma} \left(\int_{z_t}^1 (c_t(z) + i_t(z)) dz \right) + \lambda_0 [r_0 + (1 + \tau_0)(1 - \delta)] \frac{dk_0}{d\sigma} \\
&= \sum_{t=0}^{\infty} \beta^t \lambda_t R_t \frac{dm_t}{d\sigma} + \lambda_0 [r_0 + (1 + \tau_0)(1 - \delta)] \frac{dk_0}{d\sigma}.
\end{aligned}$$

The second equality follows firstly from (20) and secondly from the fact that the cash-in-advance restriction is binding. In order to get (31) one set $\frac{dk_0}{d\sigma} = 0$; to get the compensate income measure, for the situation in which long-run capital neutrality is valid, one set $\frac{dW}{d\sigma} = 0$ and $[r_0 + (1 + \tau_0)(1 - \delta)] = \beta^{-1}$. ■

A.3 Proposition 5

Proof. After substituting the first-order conditions (35), it follows from (32) that

$$\frac{dW}{d\sigma} = \int_0^{\infty} e^{-\rho t} \lambda \left[\frac{dc_1}{d\sigma} + (\pi + r) \frac{dm_1}{d\sigma} + (\pi + r + p^d) \frac{dm_2}{d\sigma} + p \frac{dc_2}{d\sigma} \right] dt. \tag{61}$$

From the goods market equilibrium, equation (48), it follows that

$$\int_0^{\infty} e^{-\rho t} \lambda \left(y_{11} \frac{dp}{d\sigma} + y_{12} \frac{dk}{d\sigma} - \frac{dc_1}{d\sigma} - \frac{d\dot{k}}{d\sigma} \right) dt = 0, \tag{62}$$

and from the liquidity services equilibrium, equation (46), it follows that

$$\begin{aligned}
0 = & \int_0^\infty e^{-\rho t} \lambda \left[q \left(y_{21} \frac{dp}{d\sigma} + y_{22} \frac{dk}{d\sigma} \right) - p \frac{dc_2}{d\sigma} - (p^d + (\pi + r)(1 - \zeta)) \frac{dm_2}{d\sigma} \right] dt \\
& + \int_0^\infty e^{-\rho t} \lambda \left[y_2 \frac{d\lambda}{d\sigma} - c_2 \frac{dp}{d\sigma} - m_2 \frac{d(p^d + (\pi + r)(1 - \zeta))}{d\sigma} \right] dt.
\end{aligned} \tag{63}$$

Adding (62) and (63) to (61), recalling (47), (39), the transversality condition, and that capital is bounded, it follows (49). ■

B Welfare Characterization

B.1 Proposition 2

Before proving proposition 2 it is useful to prove the following lemma:

Lemma 1 If the instantaneous utility is strictly concave and the transaction cost function is strictly convex, the solution path which satisfies the first-order conditions, the transversality conditions, and the market equilibrium equations, maximizes the intertemporal utility for a given path of the real quantity of money.

Proof. Let's suppose that there is another path for each variable, which satisfies the market equilibrium equations, such that utility is higher. Let's indicate it by primed variables. Consequently, it follows that

$$\begin{aligned}
0 & < \int_0^\infty e^{-\rho t} [u(c'_{1t}, s(c'_{1t}, m'_{1t}, c'_{21t})) - u(c_{1t}, s(c_{1t}, m_{1t}, c_{21t}))] dt \\
& \leq \int_0^\infty e^{-\rho t} [(u_{1t} + u_{2t}s_{1t})(c'_{1t} - c_{1t}) + u_{2t}s_{2t}(c'_{2t} - c_{2t}) + u_{2t}s_{3t}(c'_{3t} - c_{3t})] dt \\
& \quad \text{(by concavity)} \\
& = \int_0^\infty e^{-\rho t} \lambda_t [(1 + g_{1t})(c'_{1t} - c_{1t}) + (\pi_t + r_t)(c'_{2t} - c_{2t}) + p_t(c'_{3t} - c_{3t})] dt, \tag{64} \\
& \quad \text{(by the first-order conditions of the market's problem)}
\end{aligned}$$

where the unprimed variables are the solution for the market economy. But, from the market equilibrium equations, it follows that

$$0 = \int_0^\infty e^{-\rho t} \lambda \left\{ \left[y_1(p, k) + p y_1(p, k) - c_1 - g(c_1, m_2, c_{22}) - \dot{k} \right] - \left[y_1(p', k') + p' y_2(p', k') - c'_1 - g(c'_1, m'_2, c'_{22}) - \dot{k}' \right] \right\} dt,$$

which implies,³⁹ by convexity, that

$$0 \leq \int_0^\infty e^{-\rho t} \lambda \left\{ r(k' - k) - (1 + g_1)(c'_1 - c_1) - g_2(m'_2 - m_2) - g_3(c'_{22} - c_{22}) + y_2(p' - p) - c_2(p' - p) - p(c'_2 - c_2) - \frac{d}{dt}(k' - k) \right\}. \quad (65)$$

Integrating by parts the last term in (65), after recalling the transversality condition and that capital is bounded, it follows that

$$\int_0^\infty e^{-\rho t} \lambda \frac{d}{dt}(k' - k) dt = - \int_0^\infty e^{-\rho t} \lambda (k' - k) \left(-\rho + \frac{\dot{\lambda}}{\lambda} \right) dt.$$

Substituting this last equation, the first-order condition, and the Euler equation into (65), adding the result to (64), it follows that

$$\begin{aligned} 0 &< \int_0^\infty e^{-\rho t} \lambda (\pi + r)(m' - m) dt \\ &= 0, \end{aligned}$$

because the path of real quantity of money is given. The result follows by contradiction. ■

Proof. The Central Planner solves

$$\max \int_0^\infty e^{-\rho t} u(c_1, s(c_1, m_1, c_{21})) dt,$$

³⁹If instead of substituting the market equilibrium equation, it had been substituted the household's budget constraint, it would have been proved that the first-order conditions solve the household problem. See, as an example, the appendix in Cole and Kocherlakota (1998).

subject to

$$\begin{aligned}
\dot{k} &= l_1 f_1(k_1) - c_1 - g(c_1, m_2, c_{22}), \\
c_{21} + c_{22} &= l_2 f_2(k_2), \\
m &= m_1 + m_2, \quad m_t \text{ and } k_0 \text{ given}, \\
l_1 + l_2 &= 1, \text{ and} \\
l_1 k_1 + l_2 k_2 &= k,
\end{aligned}$$

where f_i is the production function of the i -th sector, l_i is the fraction of employment in the i -th sector, and k_i is capital per worker in the i -th sector. Because the Central Planner cannot choose m_t , and the first-order conditions for capital and worker allocations jointly with the production functions imply the offer's market functions, this problem is equivalent to maximizing the intertemporal utility restricting to the market equilibrium equations. The proposition follows from the lemma. ■

B.2 Proposition 4

Proof. The restricted Central Planner solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t) \quad (66)$$

subject to

$$\begin{aligned}
m_t &\geq \int_{\text{Cash Goods}} (c_t(z) + i_t(z)) dz, \\
f(k_t, n_t) - \int_{\text{Credit Goods}} (1 + R(z))(c_t(z) + i_t(z)) dz - \int_{\text{Cash Goods}} (c_t(z) + i_t(z)) dz &\geq 0, \quad (67)
\end{aligned}$$

$$k_{t+1} = i_t + (1 - \delta)k_t, \quad (68)$$

and

$$k_0, \quad m_t \text{ any } t \text{ given.} \quad (69)$$

The difficulty with this maximization problem is that the restriction (67) is not convex. However, because the transaction cost function, $R(z)$, is strictly increasing and because the transaction cost is linear for a given index, any restricted maximum for (66) will have the

following property: there will be an index, Z_t , such that every good whose index is lower than this threshold, is acquired as credit (they are the low transaction-cost goods); the other goods are acquired as cash goods (they are the high transaction-cost goods). Consequently, the Central Planner restricted problem can be rewritten as solving (66) subject to

$$m_t = \int_{Z_t}^1 (c_t(z) + i_t(z))dz, \quad (70)$$

$$f(k_t, n_t) - \int_0^{Z_t} (1 + R(z))(c_t(z) + i_t(z))dz - m_t = 0, \quad (71)$$

and (68) and (69). This is a standard concave maximization problem. Let's suppose that there is a solution, which produces a higher value for the restricted welfare than the market's solution. Let's indicate this solution by primed variables. Consequently,

$$0 < \sum_{t=0}^{\infty} \beta^t \{u(c'_t, 1 - n'_t) - u(c_t, 1 - n_t)\},$$

where the unprimed variables represent the solution for the market economy. It follows that

$$\begin{aligned} 0 &< \sum_{t=0}^{\infty} \beta^t \left\{ u_1(c_t, 1 - n_t) c_t^{\frac{1}{\theta}} \int_0^1 c_t^{-\frac{1}{\theta}} (c'_t(z) - c_t(z)) dz - u_2(c_t, 1 - n_t) (n'_t - n_t) \right\}, \\ &\quad (\text{by concavity}) \\ &= \sum_{t=0}^{\infty} \beta^t \lambda_t \left\{ \int_0^{Z_t} (1 + R(z))(c'_t(z) - c_t(z)) dz + \int_{Z_t}^1 (1 + \mu_t)(c'_t(z) - c_t(z)) dz \right. \\ &\quad \left. - u_2(c_t, 1 - n_t) (n'_t - n_t) \right\}. \\ &\quad (\text{by the first-order conditions of the market's problem}) \end{aligned} \quad (72)$$

From (71) it follows that

$$\begin{aligned} 0 &\leq \sum_{t=0}^{\infty} \beta^t \lambda_t \left\{ - \int_0^{Z_t} (1 + R(z))(c'_t(z) - c_t(z) + i'_t(z) - i_t(z)) dz \right. \\ &\quad - \int_{Z_t}^1 (1 + \mu_t)(c'_t(z) - c_t(z) + i'_t(z) - i_t(z)) dz - (1 + R(Z_t))(c_t(Z_t) + i_t(Z_t))(Z'_t - Z_t) \\ &\quad \left. + f_1(k_t, n_t)(k'_t - k_t) + f_2(k_t, n_t)(n'_t - n_t) \right\}, \end{aligned}$$

and from (70) it follows that

$$0 \leq \sum_{t=0}^{\infty} \beta^t \lambda_t (1 + \mu_t) \left\{ (c_t(z_t) + i_t(z_t))(z'_t - z_t) - \int_{z_t}^1 (1 + \mu_t)(c'_t(z) - c_t(z) + i'_t(z) - i_t(z)) dz \right\}.$$

Adding these last two inequalities to (72) it follows that

$$\begin{aligned} 0 < \sum_{t=0}^{\infty} \beta^t \lambda_t \{ f_1(k_t, n_t)(k'_t - k_t) \\ - \int_0^{z_t} (1 + R(z))(i'_t(z) - i_t(z)) dz - (1 + \mu_t) \int_{z_t}^1 (i'_t(z) - i_t(z)) dz \}. \end{aligned} \quad (73)$$

Additionally, adding the Euler equation (30), recalling (29) and that the market interest rate is equal to capital's marginal product, it is possible to write

$$\begin{aligned} \sum_{t=0}^T \beta^t \lambda_t f_1(k_t, n_t)(k'_t - k_t) &= \sum_{t=0}^T \beta^t \lambda_t q_t [k'_{t+1} - k_{t+1} - (1 - \delta)(k'_t - k_t)] - \beta^T \lambda_T q_T (k'_T - k_T) \\ &= \sum_{t=0}^T \beta^t \lambda_t q_t (i'_t - i_t) - \beta^T \lambda_T q_T (k'_T - k_T), \end{aligned}$$

where the last equality comes from the capital's accumulation equation. Substituting this last equality into (73), it follows that

$$\begin{aligned} 0 < \lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t \lambda_t \{ q_t (i'_t - i_t) \\ - \int_0^{z_t} (1 + R(z))(i'_t(z) - i_t(z)) dz - (1 + \mu_t) \int_{z_t}^1 (i'_t(z) - i_t(z)) dz \} \\ + \lim_{T \rightarrow \infty} \beta^T \lambda_T q_T (k'_T - k_T). \end{aligned} \quad (74)$$

From the aggregator function, which defines the investment good, and from the first-order

conditions (26) and (27), it follows that

$$\begin{aligned}
i'_t &\equiv \left(\int_0^1 i'_t(z)^{\frac{\theta-1}{\theta}} dz \right)^{\frac{\theta}{\theta-1}} \\
&= i_t + q_t^{-1} \int_0^{z_t} (1 + R(z))(i'_t(z) - i_t(z)) dz + q_t^{-1} (1 + \mu_t) \int_{z_t}^1 (i'_t(z) - i_t(z)) dz \\
&\quad + \text{Second-Order Terms}_t.
\end{aligned} \tag{75}$$

The remainder of the Taylor expansion of the aggregator function is negative due to concavity. Substituting (75) into (74), recalling the transversality condition and that capital is bounded, it follows that

$$0 < \sum_{t=0}^{\infty} \text{Second-Order Terms}_t \leq 0.$$

The contradiction proves the proposition. ■

B.3 Proposition 6

Proof. The Central Planner solves

$$\max \int_0^{\infty} e^{-\rho t} u(c_{1t}, s(m_{1t}, m_{2t}, c_{2t})) dt,$$

subject to

$$0 = g(c_{2t}, m_{2t}) - \frac{1}{L} F_2(L_{2t}, K_{2t}), \tag{76}$$

$$\dot{k} = \frac{1}{L} F_1(L_{1t}, K_{1t}) - c_{1t}, \tag{77}$$

$$L_{1t} + L_{2t} = L, \tag{78}$$

$$K_{1t} + K_{2t} = K_t, \tag{79}$$

$$K_0 \text{ and } b_t \equiv m_{1t} + \zeta m_{2t} \text{ given.} \tag{80}$$

Let's suppose that there is a solution, which produces a higher value for the restricted welfare than the market's solution. Let's indicate this solution by primed variables. Conse-

quently,

$$\begin{aligned}
0 &< \int_0^\infty e^{-\rho t} \{u(c'_{1t}, s(m'_{1t}, m'_{2t}, c'_{2t})) - u(c_{1t}, s(m_{1t}, m_{2t}, c_{2t}))\} dt, \\
&\leq \int_0^\infty e^{-\rho t} \{u_1(c'_{1t} - c_{1t}) + u_2 s_1(m'_{1t} - m_{1t}) + u_2 s_2(m'_{2t} - m_{2t}) + u_2 s_3(c'_{2t} - c_{2t})\} dt, \\
&\quad (\text{by concavity}) \\
&= \int_0^\infty e^{-\rho t} \lambda_t \{(c'_{1t} - c_{1t}) + R_t(m'_{1t} - m_{1t}) + (p_t^d + R_t)(m'_{2t} - m_{2t}) + p_t(c'_{2t} - c_{2t})\} dt. \\
&\quad (\text{by the first-order conditions (35)-(38)}) \tag{81}
\end{aligned}$$

From (76), it follows that

$$\begin{aligned}
0 &\leq \int_0^\infty e^{-\rho t} \lambda_t q_t \left\{ \frac{1}{L} [F_{21}(L'_2 - L_2) + F_{22}(K'_2 - K_2)] - g_1(c'_{2t} - c_{2t}) - g_2(m'_{2t} - m_{2t}) \right\} dt, \\
&\quad (\text{by convexity}) \\
&= \int_0^\infty e^{-\rho t} \lambda_t q_t \left\{ \frac{1}{L} [F_{21}(L'_2 - L_2) + F_{22}(K'_2 - K_2)] - p_t(c'_{2t} - c_{2t}) \right. \\
&\quad \left. - (p_t^d + R_t(1 - \zeta))(m'_{2t} - m_{2t}) \right\} dt. \tag{82} \\
&\quad (\text{by the first-order conditions (42) and (43)})
\end{aligned}$$

From (77), it follows that

$$\begin{aligned}
0 &\leq \int_0^\infty e^{-\rho t} \lambda_t \left\{ \frac{1}{L} [F_{11}(L'_1 - L_1) + F_{12}(K'_1 - K_1)] - (c'_{1t} - c_{1t}) - \frac{d}{dt}(k'_t - k_t) \right\} dt \tag{83} \\
&\quad (\text{by convexity}) \\
&= \int_0^\infty e^{-\rho t} \lambda_t \left\{ \frac{1}{L} [F_{11}(L'_1 - L_1) + F_{12}(K'_1 - K_1)] - (c'_{1t} - c_{1t}) + \left(\frac{\dot{\lambda}_t}{\lambda_t} - \rho \right) (k'_t - k_t) \right\} dt
\end{aligned}$$

Adding (81)-(83), it follows that

$$\begin{aligned}
0 &< \int_0^\infty e^{-\rho t} \lambda_t \{ R_t [(m'_{1t} - m_{1t}) + \zeta(m'_{2t} - m_{2t})] \\
&\quad + \frac{1}{L} [F_{11}(L'_1 - L_1) + F_{12}(K'_1 - K_1)] + \frac{q_t}{L} [F_{21}(L'_2 - L_2) + F_{22}(K'_2 - K_2)] - r_t(k'_t - k_t) \} dt, \tag{84}
\end{aligned}$$

where in the last equation in (83) the Euler equation (39) was substituting.

From (78), (79), and from factor mobility across sector, it follows that

$$\frac{1}{L} [F_{11}(L'_1 - L_1) + F_{12}(K'_1 - K_1)] + \frac{q_t}{L} [F_{21}(L'_2 - L_2) + F_{22}(K'_2 - K_2)] = r_t(k'_t - k_t).$$

Substituting this last equation into (84), it follows that

$$\begin{aligned} 0 &< \int_0^\infty e^{-\rho t} \lambda_t R_t [(m'_{1t} - m_{1t}) + \zeta(m'_{2t} - m_{2t})] dt \\ &= \int_0^\infty e^{-\rho t} \lambda_t R_t (b'_t - b_t) dt \\ &= 0. \text{ (by (80))} \end{aligned}$$

The result follows by contradiction. ■